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# On multiple Bernoulli polynomials and multiple $L$ -functions of root systems (Analytic Number Theory and Related Areas)

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CITATION:

Komori, Yasushi ...[et al]. On multiple Bernoulli polynomials and multiple  $L$ -functions of root systems (Analytic Number Theory and Related Areas). 数理解析研究所講究録 2009, 1665: 114-126

ISSUE DATE:

2009-10

URL:

<http://hdl.handle.net/2433/141047>

RIGHT:

# On multiple Bernoulli polynomials and multiple $L$ -functions of root systems

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## §1. Introduction: Review of Classical Theory

In this article we propose generalizations of Bernoulli polynomials and  $L$ -functions associated with root systems. To state our results, first we recall the classical theory for the Riemann zeta-function and Bernoulli numbers.

The following is a well-known formula for the Riemann zeta-function and Bernoulli numbers.

For  $k \in \mathbb{Z}_{\geq 1}$ ,

$$2\zeta(2k) = -B_{2k} \frac{(2\pi i)^{2k}}{(2k)!},$$

where

$$\frac{te^t}{e^t - 1} = - \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

By using this formula, we obtain for  $k \in \mathbb{Z}_{\geq 1}$ ,

$$\zeta(2k) + (-1)^{2k} \zeta(2k) = -B_{2k} \frac{(2\pi i)^{2k}}{(2k)!},$$

$$\zeta(2k+1) + (-1)^{2k+1} \zeta(2k+1) = -B_{2k+1} \frac{(2\pi i)^{2k+1}}{(2k+1)!} = 0.$$

Hence we have important relations:

For  $k \in \mathbb{Z}_{\geq 2}$ ,

$$\zeta(k) + (-1)^k \zeta(k) = -B_k \frac{(2\pi i)^k}{k!},$$

value-relations = Bernoulli numbers.

This procedure can be applied to Lerch zeta-functions and periodic Bernoulli functions. Let  $\varphi(s, y)$  be the Lerch zeta-function defined by

$$\varphi(s, y) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n y}}{n^s}.$$

Then a formula for Lerch zeta-functions implies

For  $k \in \mathbb{Z}_{\geq 2}$  and  $y \in \mathbb{R}$ ,

$$\begin{aligned} \varphi(k, y) + (-1)^k \varphi(k, -y) &= -B_k(\{y\}) \frac{(2\pi i)^k}{k!}, \\ \text{functional relations} &= \text{periodic Bernoulli functions.} \end{aligned}$$

Here

$$\frac{te^{t\{y\}}}{e^t - 1} = - \sum_{k=0}^{\infty} B_k(\{y\}) \frac{t^k}{k!},$$

and  $\{y\} = y - [y]$  (i.e. fractional part).

Once we obtain periodic Bernoulli functions, we can calculate special values of  $L$ -functions.

For a primitive character  $\chi$  of conductor  $f$  and  $k \in \mathbb{Z}_{\geq 2}$  satisfying  $(-1)^k \chi(-1) = 1$ , we have

$$\begin{aligned} L(k, \chi) &= \sum_{n=1}^{\infty} \frac{\chi(n)}{n^k} \\ &= \frac{(-1)^{k+1}}{2} \frac{(2\pi i)^k}{k! f^k} g(\chi) B_{k, \bar{\chi}}, \end{aligned}$$

where  $g(\chi)$  is the Gauss sum and

$$B_{k, \chi} = f^{k-1} \sum_{a=1}^f \chi(a) B_k(a/f).$$

Our aim is to find a good class of multiple zeta-functions which generalize the theory above.

## §2. Overview of Our Results

Based on the observation given in the previous section, we will construct multiple generalizations of Bernoulli polynomials and multiple  $L$ -functions associated with arbitrary root systems. Before introducing the general theory, we give two simple theorems by using the explicit form of the root system of type  $A_2$ .

For  $s_1, s_2, s_3 \in \mathbb{C}$  and  $y_1, y_2 \in \mathbb{R}$ , we consider the convergent series

$$\zeta_2(s_1, s_2, s_3, y_1, y_2; A_2) = \sum_{m, n=1}^{\infty} \frac{e^{2\pi i(m y_1 + n y_2)}}{m^{s_1} n^{s_2} (m+n)^{s_3}}.$$

**Theorem A.** For  $k_1, k_2, k_3 \in \mathbb{Z}_{\geq 2}$ ,

$$\begin{aligned} & \zeta_2(k_1, k_2, k_3, y_1, y_2; A_2) + (-1)^{k_1} \zeta_2(k_1, k_3, k_2, -y_1 + y_2, y_2; A_2) \\ & + (-1)^{k_2} \zeta_2(k_3, k_2, k_1, y_1, y_1 - y_2; A_2) + (-1)^{k_2+k_3} \zeta_2(k_3, k_1, k_2, -y_1 + y_2, -y_1; A_2) \\ & + (-1)^{k_1+k_3} \zeta_2(k_2, k_3, k_1, -y_2, y_1 - y_2; A_2) + (-1)^{k_1+k_2+k_3} \zeta_2(k_2, k_1, k_3, -y_2, -y_1; A_2) \\ & = (-1)^3 P(k_1, k_2, k_3, y_1, y_2; A_2) \frac{(2\pi i)^{k_1+k_2+k_3}}{k_1! k_2! k_3!}, \end{aligned}$$

where  $P(k_1, k_2, k_3, y_1, y_2; A_2)$  is a multiple periodic Bernoulli function (defined later). In particular, we have

$$\zeta_2(2, 2, 2, 0, 0; A_2) = \frac{1}{6} (-1)^3 \frac{1}{3780} \frac{(2\pi i)^{2+2+2}}{2!2!2!} = \frac{\pi^6}{2835}.$$

cf.

$$\varphi(k, y) + (-1)^k \varphi(k, -y) = -B_k(\{y\}) \frac{(2\pi i)^k}{k!}, \quad \zeta(2) = \frac{1}{2} (-1) \frac{1}{6} \frac{(2\pi i)^2}{2!} = \frac{\pi^2}{6}.$$

For  $s_1, s_2, s_3 \in \mathbb{C}$  and primitive Dirichlet characters  $\chi_1, \chi_2, \chi_3$ , consider the convergent series

$$L_2(s_1, s_2, s_3, \chi_1, \chi_2, \chi_3; A_2) = \sum_{m, n=1}^{\infty} \frac{\chi_1(m) \chi_2(n) \chi_3(m+n)}{m^{s_1} n^{s_2} (m+n)^{s_3}}.$$

**Theorem B.** For  $k \in \mathbb{Z}_{\geq 2}$  and a primitive Dirichlet character  $\chi$  of conductor  $f$  such that  $(-1)^k \chi(-1) = 1$ ,

$$L_2(k, k, k, \chi, \chi, \chi; A_2) = \frac{(-1)^{3k+3}}{6} \left( \frac{(2\pi i)^k}{k! f^k} g(\chi) \right)^3 B_{k, k, k, \bar{\chi}, \bar{\chi}, \bar{\chi}}(A_2),$$

where  $B_{k_1, k_2, k_3, \chi_1, \chi_2, \chi_3}(A_2)$  is a multiple generalized Bernoulli number (defined later). In particular, for  $\rho_5 : \rho_5(1) = \rho_5(4) = 1, \rho_5(2) = \rho_5(3) = -1$ , we have

$$L_2(2, 2, 2, \rho_5, \rho_5, \rho_5; A_2) = \frac{(-1)^{6+3}}{6} \left( \frac{(2\pi i)^2}{2! 5^2} \sqrt{5} \right)^3 \left( -\frac{28}{125} \right) = -\frac{112 \sqrt{5}}{1171875} \pi^6.$$

cf.

$$L(k, \chi) = \frac{(-1)^{k+1}}{2} \frac{(2\pi i)^k}{k! f^k} g(\chi) B_{k, \bar{\chi}}, \quad L(2, \rho_5) = \frac{(-1)^{2+1}}{2} \frac{(2\pi i)^2}{2! 5^2} \sqrt{5} \frac{4}{5} = \frac{4 \sqrt{5}}{125} \pi^2.$$

Theorems A and B are special cases of our main theorems. In the following sections, we will formulate these facts.

### §3. Root Systems

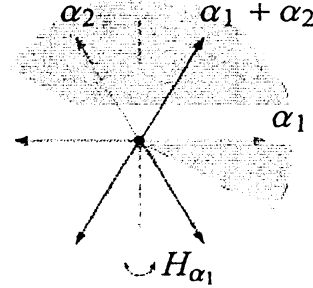
For reader's convenience, we give the definition and several examples of root systems.

### §§3.1. Definitions

Let  $V$  be an  $r$  dimensional real vector space equipped with inner product  $\langle \cdot, \cdot \rangle$ .

A root system  $\Delta \subset V$  is a set of vectors (roots):

- (1)  $|\Delta| < \infty$  and  $0 \notin \Delta$ ,
- (2)  $\sigma_\alpha \Delta = \Delta$  for all  $\alpha \in \Delta$ ,
- (3)  $\langle \alpha^\vee, \beta \rangle \in \mathbb{Z}$  for all  $\alpha, \beta \in \Delta$ ,
- (4)  $\alpha, c\alpha \in \Delta \implies c = \pm 1$ ,



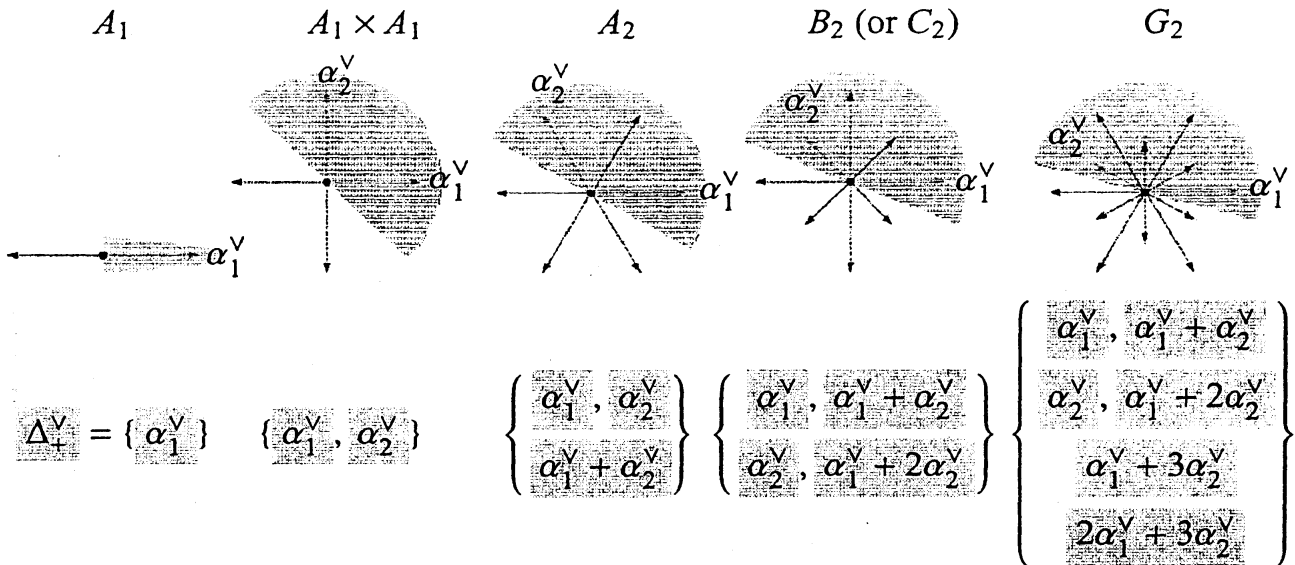
where  $\sigma_\alpha$  denotes the reflection with respect to the hyperplane  $H_\alpha$  orthogonal to  $\alpha$  and  $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$  (coroot).

Let  $W$  be the Weyl group (the group generated by all  $\sigma_\alpha$ ). Let  $\{\alpha_1, \dots, \alpha_r\}$  be fundamental roots (a basis s.t.  $\alpha = c_1\alpha_1 + \dots + c_r\alpha_r \in \Delta$  with all  $c_i \geq 0$  or  $c_i \leq 0$ ). Let  $\Delta_+$  be positive roots (all roots  $\alpha = c_1\alpha_1 + \dots + c_r\alpha_r \in \Delta$  with all  $c_i \geq 0$ ) and  $P_{++}$ , strictly dominant weights ( $= \bigoplus \mathbb{Z}_{\geq 1}\lambda_i$ ,  $\{\lambda_1, \dots, \lambda_r\}$  dual basis of  $\{\alpha_1^\vee, \dots, \alpha_r^\vee\}$ ). The key fact which plays an essential role is that the nice group  $W$  acts on  $\Delta$ .

### §§3.2. Examples

Since we mainly treat coroots, we give examples of root systems in terms of coroots. Note that if  $\Delta$  is a root system, then  $\Delta^\vee = \{\alpha^\vee \mid \alpha \in \Delta\}$  is also a root system.

There is only one root system of rank 1 and there are four root systems of rank 2:



In this article, we use these root systems in examples for simplicity. It should be noted that root systems are classified as  $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$  and our theory can be applied to all these root systems.

## §4. Zeta-Functions of Root Systems

### §§4.1. Witten Zeta-Functions

As prototypes of zeta-functions of root systems, we give the definition of Witten zeta-functions, which were originally introduced to calculate the volumes of certain moduli spaces.

Witten zeta-functions ([13, 14]): For a complex simple Lie algebra  $\mathfrak{g}$  of type  $X_r$ ,

$$\zeta_W(s; X_r) = \sum_{\varphi} (\dim \varphi)^{-s} = K(X_r)^s \sum_{\lambda \in P_{++}} \prod_{\alpha \in \Delta_+} \frac{1}{\langle \alpha^\vee, \lambda \rangle^s},$$

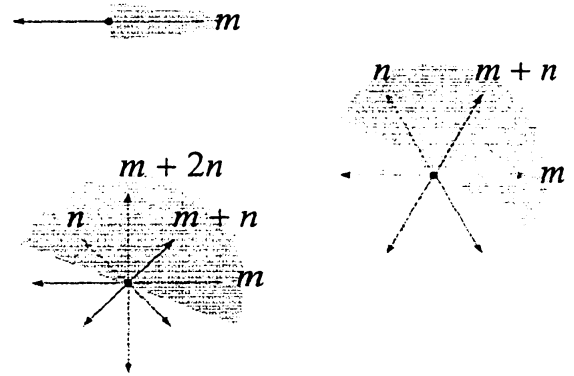
where the summation runs over all finite dimensional irreducible representations  $\varphi$  and  $K(X_r) \in \mathbb{Z}_{\geq 1}$  is a constant.

From the second expression of the definition, we see that the explicit forms of Witten zeta-functions are obtained by formally replacing  $\alpha_1^\vee$  and  $\alpha_2^\vee$  by  $m$  and  $n$  respectively:

$$\zeta_W(s; A_1) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \zeta(s),$$

$$\zeta_W(s; A_2) = 2^s \sum_{m,n=1}^{\infty} \frac{1}{m^s n^s (m+n)^s},$$

$$\zeta_W(s; B_2) = 6^s \sum_{m,n=1}^{\infty} \frac{1}{m^s n^s (m+n)^s (m+2n)^s}.$$



### §§4.2. Zeta-Functions of Root Systems

**Definition 1** ([6; 7, 8, 12]). Zeta-functions of root systems: For a root system  $\Delta$  of type  $X_r$ , define

$$\zeta_r(s, y; X_r) = \sum_{\lambda \in P_{++}} e^{2\pi i \langle y, \lambda \rangle} \prod_{\alpha \in \Delta_+} \frac{1}{\langle \alpha^\vee, \lambda \rangle^{s_\alpha}},$$

where  $s = (s_\alpha)_{\alpha \in \Delta_+} \in \mathbb{C}^{|\Delta_+|}$  and  $y \in V$ .

To define an action of the Weyl group, we extend  $s = (s_\alpha)_{\alpha \in \Delta_+}$  to  $(s_\alpha)_{\alpha \in \Delta}$  by  $s_\alpha = s_{-\alpha}$  and define  $(ws)_\alpha = s_{w^{-1}\alpha}$ . Then we have our first theorem.

**Theorem 1** ([8]). For  $s = k = (k_\alpha)_{\alpha \in \Delta_+} \in \mathbb{Z}_{\geq 2}^{|\Delta_+|}$ , we have

$$\sum_{w \in W} \left( \prod_{\alpha \in \Delta_+ \cap w^{-1}\Delta_-} (-1)^{k_\alpha} \right) \zeta_r(w^{-1}k, w^{-1}y; X_r) = (-1)^{|\Delta_+|} P(k, y; X_r) \left( \prod_{\alpha \in \Delta_+} \frac{(2\pi i)^{k_\alpha}}{k_\alpha!} \right),$$

where  $P(k, y; X_r)$  is a multiple periodic Bernoulli function (defined later).

cf.  $(X_r = A_1)$

$$\varphi(k, y) + (-1)^k \varphi(k, -y) = -B_k(\{y\}) \frac{(2\pi i)^k}{k!} \quad (W = \{\text{id}, \sigma_\alpha\}).$$

## §5. Special Zeta-Values

Theorem 1 directly implies the following theorem:

**Theorem 2** ([8]). For  $\mathbf{k} = (k_\alpha)_{\alpha \in \Delta_+} \in (2\mathbb{Z}_{\geq 1})^{|\Delta_+|}$  satisfying  $w^{-1}\mathbf{k} = \mathbf{k}$  for all  $w \in W$ ,

$$\zeta_r(\mathbf{k}, \mathbf{0}; X_r) = \frac{(-1)^{|\Delta_+|}}{|W|} P(\mathbf{k}, \mathbf{0}; X_r) \left( \prod_{\alpha \in \Delta_+} \frac{(2\pi i)^{k_\alpha}}{k_\alpha!} \right) \in \mathbb{Q} \pi^{\sum_{\alpha \in \Delta_+} k_\alpha}.$$

cf.  $(X_r = A_1)$

$$\zeta(k) = \frac{-1}{2} B_k \frac{(2\pi i)^k}{k!} \in \mathbb{Q} \pi^k \quad (k \in 2\mathbb{Z}_{\geq 1}).$$

In particular,  $\mathbf{k} = (k)_{\alpha \in \Delta_+}$  with  $k \in 2\mathbb{Z}_{\geq 1}$  (that is, all  $k_\alpha = k$ ) satisfies the condition in Theorem 2. In this case,  $\zeta_r(\mathbf{k}, \mathbf{0}; X_r) \in \mathbb{Q} \pi^{|\Delta_+|k}$  was shown by Witten and Zagier. Our statement is a true generalization of their results since we also have for example,

$$\begin{aligned} \zeta_2((2, 4, 4, 2), \mathbf{0}; B_2) &= \sum_{m, n=1}^{\infty} \frac{1}{m^2 n^4 (m+n)^4 (m+2n)^2} \\ &= \frac{(-1)^4}{2^2 2!} \frac{53}{1513512000} \left( \frac{(2\pi i)^2}{2!} \right)^2 \left( \frac{(2\pi i)^4}{4!} \right)^2 \\ &= \frac{53\pi^{12}}{6810804000}. \end{aligned}$$

## §6. Multiple Periodic Bernoulli Functions

In this section, we give the definitions of generating functions of multiple periodic Bernoulli functions. Let  $\mathcal{V}$  be the set of all bases  $\mathbf{V} \subset \Delta_+$ ,  $\mathbf{V}^* = \{\mu_\beta^{\mathbf{V}}\}_{\beta \in \mathbf{V}}$ , the dual basis of  $\mathbf{V}^\vee = \{\beta^\vee\}_{\beta \in \mathbf{V}}$ . Let  $Q^\vee = \bigoplus_{i=1}^r \mathbb{Z} \alpha_i^\vee$  be the coroot lattice and  $L(\mathbf{V}^\vee) = \bigoplus_{\beta \in \mathbf{V}} \mathbb{Z} \beta^\vee$ , which is a sublattice of  $Q^\vee$  with finite index ( $|Q^\vee/L(\mathbf{V}^\vee)| < \infty$ ).

Fix a certain  $\phi \in V$  and define a multiple generalization of fractional part as

$$\{y\}_{\mathbf{V}, \beta} = \begin{cases} \{\langle y, \mu_\beta^{\mathbf{V}} \rangle\} & (\langle \phi, \mu_\beta^{\mathbf{V}} \rangle > 0), \\ 1 - \{-\langle y, \mu_\beta^{\mathbf{V}} \rangle\} & (\langle \phi, \mu_\beta^{\mathbf{V}} \rangle < 0). \end{cases}$$

By using these definitions, we have

**Definition 2** (generating function [8, 9, 10]). For  $\mathbf{t} = (t_\alpha)_{\alpha \in \Delta_+}$ ,

$$\begin{aligned} F(\mathbf{t}, \mathbf{y}; X_r) &= \sum_{\mathbf{V} \in \mathcal{V}} \left( \prod_{\gamma \in \Delta_+ \setminus \mathbf{V}} \frac{t_\gamma}{t_\gamma - \sum_{\beta \in \mathbf{V}} t_\beta \langle \gamma^\vee, \mu_\beta^{\mathbf{V}} \rangle} \right) \\ &\quad \times \frac{1}{|Q^\vee/L(\mathbf{V}^\vee)|} \sum_{q \in Q^\vee/L(\mathbf{V}^\vee)} \left( \prod_{\beta \in \mathbf{V}} \frac{t_\beta \exp(t_\beta \{y + q\}_{\mathbf{V}, \beta})}{e^{t_\beta} - 1} \right). \end{aligned}$$

**Definition 3** (multiple periodic Bernoulli functions [8, 9, 10]).

$$F(\mathbf{t}, \mathbf{y}; X_r) = \sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^{|\Delta_+|}} P(\mathbf{k}, \mathbf{y}; X_r) \prod_{\alpha \in \Delta_+} \frac{t_\alpha^{k_\alpha}}{k_\alpha!}.$$

cf.  $(X_r = A_1)$

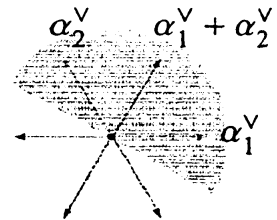
$$F(t, y) = \frac{te^{t\{y\}}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(\{y\}) \frac{t^k}{k!}.$$

## §7. Example: $A_2$ Case

We calculate a multiple periodic Bernoulli function and its generating function in the case of the root system of type  $A_2$ .

We have the basic data as follows:

$$\begin{aligned} \Delta_+^\vee &= \{\alpha_1^\vee, \alpha_2^\vee, \alpha_1^\vee + \alpha_2^\vee\}, \quad \mathcal{V} = \{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3\}, \\ \mathbf{t} &= (t_{\alpha_1}, t_{\alpha_2}, t_{\alpha_1 + \alpha_2}) = (t_1, t_2, t_3), \\ \mathbf{y} &= y_1 \alpha_1^\vee + y_2 \alpha_2^\vee. \end{aligned}$$



Fix a sufficiently small  $\varepsilon > 0$  and  $\phi = \alpha_1^\vee + \varepsilon \alpha_2^\vee$ . Then by using these data, we have the generating function and a multiple periodic Bernoulli function as

$$F(\mathbf{t}, \mathbf{y}; A_2) =$$

basis  $\mathbf{V} \subset \Delta_+$ , dual basis  $\mathbf{V}^*$

$$\frac{t_3}{t_3 - t_1 - t_2} \frac{t_1 e^{t_1 \{y_1\}}}{e^{t_1} - 1} \frac{t_2 e^{t_2 \{y_2\}}}{e^{t_2} - 1}$$

$$(\mathbf{V}_1^\vee = \{\alpha_1^\vee, \alpha_2^\vee\}, \mathbf{V}_1^* = \{\lambda_1, \lambda_2\})$$

$$+ \frac{t_2}{t_2 + t_1 - t_3} \frac{t_1 e^{t_1 \{y_1 - y_2\}}}{e^{t_1} - 1} \frac{t_3 e^{t_3 \{y_2\}}}{e^{t_3} - 1}$$

$$(\mathbf{V}_2^\vee = \{\alpha_1^\vee, \alpha_1^\vee + \alpha_2^\vee\}, \mathbf{V}_2^* = \{\lambda_1 - \lambda_2, \lambda_2\})$$

$$+ \frac{t_1}{t_1 + t_2 - t_3} \frac{t_2 e^{t_2 (1 - \{y_1 - y_2\})}}{e^{t_2} - 1} \frac{t_3 e^{t_3 \{y_1\}}}{e^{t_3} - 1}$$

$$(\mathbf{V}_3^\vee = \{\alpha_2^\vee, \alpha_1^\vee + \alpha_2^\vee\}, \mathbf{V}_3^* = \{\lambda_2 - \lambda_1, \lambda_1\})$$

For  $\mathbf{k} = \mathbf{2} = (2, 2, 2)$ ,

$$P(\mathbf{2}, \{y_1, y_2\}; A_2) = \frac{1}{3780} + \frac{1}{90}(\{y_1\} - \{y_1 - y_2\} - \{y_2\})$$

...

$$\begin{aligned} &+ \frac{1}{30}(-\{y_1\}^6 + 4\{y_1 - y_2\}\{y_1\}^5 - 5\{y_1 - y_2\}^2\{y_1\}^4 \\ &\quad - \{y_2\}^6 - 4\{y_1 - y_2\}\{y_2\}^5 - 5\{y_1 - y_2\}^2\{y_2\}^4). \end{aligned}$$

We have a functional relation corresponding to this multiple periodic Bernoulli function:



$$\begin{aligned} & \zeta_2(2, (y_1, y_2); A_2) + \zeta_2(2, (-y_1 + y_2, y_2); A_2) + \zeta_2(2, (y_1, y_1 - y_2); A_2) \\ & + \zeta_2(2, (-y_2, y_1 - y_2); A_2) + \zeta_2(2, (-y_1 + y_2, -y_1); A_2) + \zeta_2(2, (-y_2, -y_1); A_2) \\ & = (-1)^3 P(2, (y_1, y_2); A_2) \frac{(2\pi i)^6}{(2!)^3}. \end{aligned}$$

In particular if  $(y_1, y_2) = (0, 0)$ , then

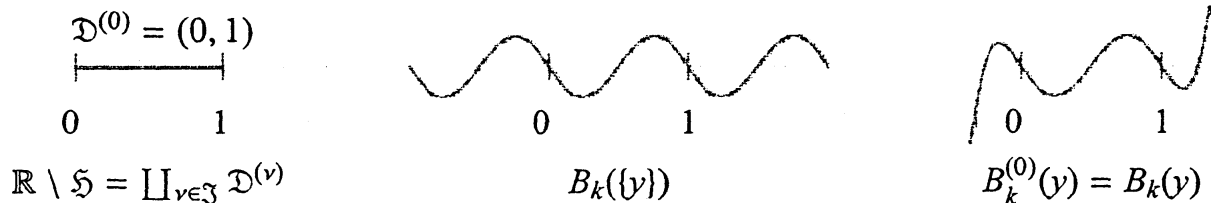
$$\zeta_2(2, (0, 0); A_2) = \frac{1}{6}(-1)^3 \frac{1}{3780} \frac{(2\pi i)^6}{(2!)^3} = \frac{\pi^6}{2835}.$$

cf.  $(X_r = A_1)$

$$\zeta(2) = \frac{1}{2}(-1) \frac{1}{6} \frac{(2\pi i)^2}{2!} = \frac{\pi^2}{6}, \quad B_2(\{y\}) = \frac{1}{6} - \{y\} + \{y\}^2.$$

## §8. Multiple Bernoulli Polynomials

In the classical theory, Bernoulli polynomials can be derived by the analytic continuation of periodic Bernoulli functions. We explain this fact. Let  $\mathfrak{H} = \{y \in \mathbb{R} \mid \{y\} \in \mathbb{Z}\} = \mathbb{Z}$  (discontinuous points of  $\{y\}$ ). Let  $\mathbb{R} \setminus \mathfrak{H} = \coprod_{v \in \mathbb{Z}} \mathcal{D}^{(v)}$ , where  $\mathcal{D}^{(v)} = (v, v+1)$ . From each  $\mathcal{D}^{(v)}$  to  $\mathbb{C}$ , the function  $B(\{y\})$  is analytically continued to a polynomial function  $B_k^{(v)}(y) = B_k(y - v) \in \mathbb{Q}[y]$ .



A similar procedure works well in general cases and we can define multiple generalizations of Bernoulli polynomials.

Let

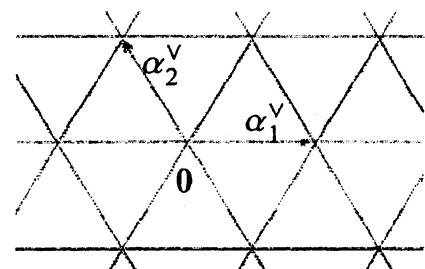
$$\mathfrak{H} = \bigcup_{V \in \mathcal{V}} \bigcup_{q \in Q^V} \bigcup_{\beta \in V} \{y \in V \mid \{y + q\}_{V, \beta} \in \mathbb{Z}\}$$

(discontinuous points of  $\{y + q\}_{V, \beta}$  appearing in the generating function).

Let

$$V \setminus \mathfrak{H} = \coprod_{v \in \mathfrak{J}} \mathcal{D}^{(v)},$$

where  $\mathcal{D}^{(v)}$  is an open connected component,  $\mathfrak{J}$  is a set of indices.

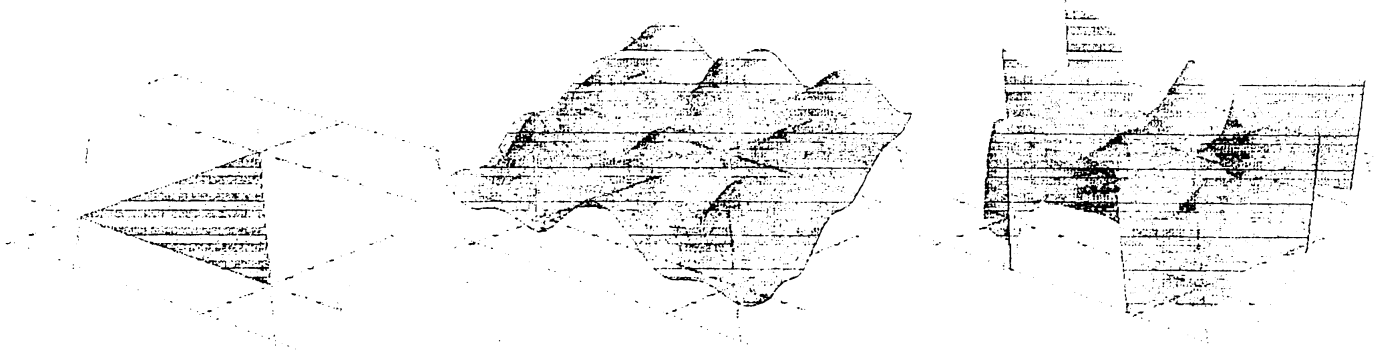


$A_2$  case

**Theorem 3** ([8, 9, 10]). From each region  $\mathcal{D}^{(v)}$  to the whole space  $\mathbb{C} \otimes V$ ,  $P(\mathbf{k}, \mathbf{y}; X_r)$  is analytically continued in  $\mathbf{y}$  to a polynomial function  $B_{\mathbf{k}}^{(v)}(\mathbf{y}; X_r) \in \mathbb{Q}[\mathbf{y}]$  of total degree at most  $|\mathbf{k}| = \sum_{\alpha \in \Delta_+} k_{\alpha}$ , where  $\mathbf{y} = \sum_{n=1}^r y_n \alpha_n^V$ .

### §§8.1. Example: $A_2$ Case

The Bernoulli polynomial  $B_2^{(0)}(\mathbf{y}; A_2)$  is obtained by the analytic continuation of the periodic Bernoulli function  $P(2, \mathbf{y}; A_2)$  from the region  $\mathcal{D}^{(0)}$ .



$$V \setminus \mathcal{H} = \coprod_{v \in \mathcal{J}} \mathcal{D}^{(v)}$$

$$P(2, \mathbf{y}; A_2)$$

(Periodic Bernoulli function)

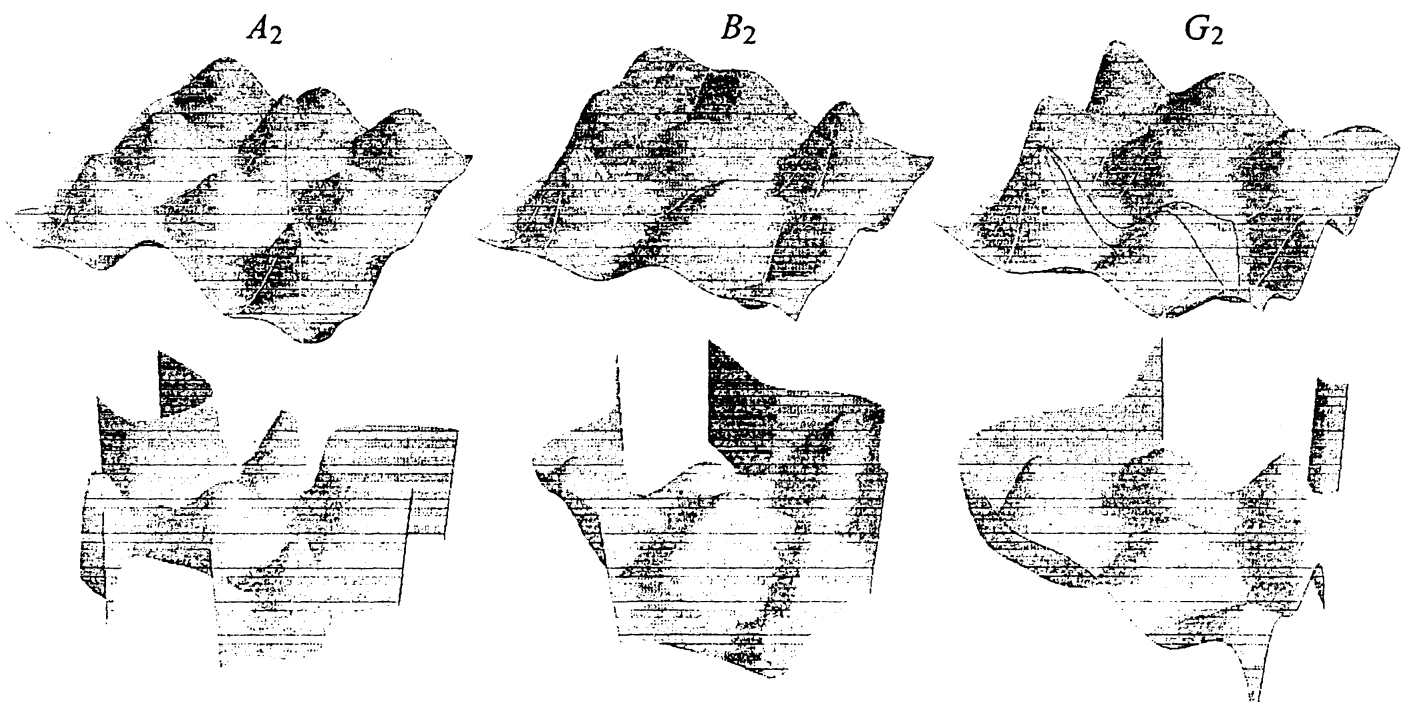
$$B_2^{(0)}(\mathbf{y}; A_2)$$

(Bernoulli polynomial)

The explicit form of the Bernoulli polynomial  $B_2^{(0)}(\mathbf{y}; A_2)$  is given as follows:

$$\begin{aligned} B_2^{(0)}(\mathbf{y}; A_2) = & \frac{1}{3780} + \frac{1}{45}(y_1 y_2 - y_1^2 - y_2^2) + \frac{1}{18}(3y_1 y_2^2 - 3y_1^2 y_2 + 2y_1^3) \\ & + \frac{1}{9}(-2y_1 y_2^3 - 3y_1^2 y_2^2 + 4y_1^3 y_2 - 2y_1^4 + y_2^4) \\ & + \frac{1}{30}(-5y_1 y_2^4 + 10y_1^2 y_2^3 + 10y_1^3 y_2^2 - 15y_1^4 y_2 + 6y_1^5) \\ & + \frac{1}{30}(6y_1 y_2^5 - 5y_1^2 y_2^4 - 5y_1^4 y_2^2 + 6y_1^5 y_2 - 2y_1^6 - 2y_2^6) \in \mathbb{Q}[\mathbf{y}]. \end{aligned}$$

### §§8.2. Further Examples: $A_2, B_2, G_2$ Cases



The graphs in the upper (resp. lower) row are those of periodic Bernoulli functions (resp. Bernoulli polynomials).

We summarize what we have obtained: we have constructed periodic Bernoulli functions so that they describe functional-relations of multiple zeta-functions of root systems, which can be calculated by using the generating function; Bernoulli polynomials are obtained by the analytic continuation of periodic Bernoulli functions.

$$\sum_{w \in W} \left( \prod_{\alpha \in \Delta_+ \cap w^{-1}\Delta_-} (-1)^{k_\alpha} \right) \zeta_r(w^{-1}\mathbf{k}, w^{-1}\mathbf{y}; X_r) = (-1)^{|\Delta_+|} P(\mathbf{k}, \mathbf{y}; X_r) \left( \prod_{\alpha \in \Delta_+} \frac{(2\pi i)^{k_\alpha}}{k_\alpha!} \right),$$

$$F(\mathbf{t}, \mathbf{y}; X_r) = \sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^{|\Delta_+|}} P(\mathbf{k}, \mathbf{y}; X_r) \prod_{\alpha \in \Delta_+} \frac{t_\alpha^{k_\alpha}}{k_\alpha!},$$

$$P(\mathbf{k}, \mathbf{y}; X_r) \iff B_{\mathbf{k}}^{(y)}(\mathbf{y}; X_r) \in \mathbb{Q}[\mathbf{y}].$$

## §9. $L$ -Functions of Root Systems

We give an application of periodic Bernoulli functions or equivalently Bernoulli polynomials. For this purpose, we define an  $L$ -analogue of zeta-functions of root systems.

**Definition 4** ([9, 10]).  $L$ -functions of root systems: For a root system  $\Delta$  of type  $X_r$ , define

$$L_r(\mathbf{s}, \chi; X_r) = \sum_{\lambda \in P_{++}} \prod_{\alpha \in \Delta_+} \frac{\chi_\alpha(\langle \alpha^\vee, \lambda \rangle)}{\langle \alpha^\vee, \lambda \rangle^{s_\alpha}},$$

where  $\chi = (\chi_\alpha)_{\alpha \in \Delta_+}$  is a set of primitive Dirichlet characters of conductors  $f_\alpha \in \mathbb{Z}_{\geq 1}$ .

We extend  $\chi = (\chi_\alpha)_{\alpha \in \Delta_+}$  to  $(\chi_\alpha)_{\alpha \in \Delta}$  by  $\chi_\alpha = \chi_{-\alpha}$  and define  $(w\chi)_\alpha = \chi_{w^{-1}\alpha}$ . Then we have value-relations of  $L$ -functions.

**Theorem 4** ([9, 10]). For  $\mathbf{s} = \mathbf{k} = (k_\alpha)_{\alpha \in \Delta_+} \in \mathbb{Z}_{\geq 2}^{|\Delta_+|}$ ,

$$\begin{aligned} \sum_{w \in W} \left( \prod_{\alpha \in \Delta_+ \cap w^{-1}\Delta_-} (-1)^{k_\alpha} \chi_\alpha(-1) \right) L_r(w^{-1}\mathbf{k}, w^{-1}\chi; X_r) \\ = (-1)^{|\Delta_+|} \left( \prod_{\alpha \in \Delta_+} \chi_\alpha(-1) g(\chi_\alpha) \frac{(2\pi i)^{k_\alpha}}{k_\alpha! f^{k_\alpha}} \right) B_{\mathbf{k}, \bar{\chi}}(X_r), \end{aligned}$$

where  $B_{\mathbf{k}, \bar{\chi}}(X_r)$  is a multiple generalized Bernoulli number (defined later).

cf.  $(X_r = A_1)$

$$L(k, \chi) + (-1)^k \chi(-1) L(k, \chi) = -\chi(-1) g(\chi) \frac{(2\pi i)^k}{k! f^k} B_{k, \bar{\chi}}.$$

## §10. Special $L$ -Values

Theorem 4 directly implies a formula for special values of  $L$ -functions:

**Theorem 5** ([9, 10]). For  $\mathbf{k} \in (\mathbb{Z}_{\geq 2})^{|\Delta_+|}$  and  $\chi$  s.t.  $w^{-1}\mathbf{k} = \mathbf{k}$ ,  $w^{-1}\chi = \chi$  for all  $w \in W$  and  $(-1)^{k_\alpha}\chi_\alpha(-1) = 1$  for all  $\alpha \in \Delta_+$ ,

$$L_r(\mathbf{k}, \chi; X_r) = \frac{(-1)^{|\mathbf{k}|+|\Delta_+|}}{|W|} \left( \prod_{\alpha \in \Delta_+} \frac{(2\pi i)^{k_\alpha}}{k_\alpha! f_\alpha^{k_\alpha}} g(\chi_\alpha) \right) B_{\mathbf{k}, \bar{\chi}}(X_r).$$

cf.  $(X_r = A_1)$

$$L(k, \chi) = \frac{(-1)^{k+1}}{2} \frac{(2\pi i)^k}{k! f^k} g(\chi) B_{k, \bar{\chi}}.$$

As an example, let  $\rho_7$  be the Dirichlet character of conductor 7 defined by  $\rho_7(1) = \rho_7(6) = 1$ ,  $\rho_7(2) = \rho_7(5) = e^{2\pi i/3}$ ,  $\rho_7(3) = \rho_7(4) = e^{4\pi i/3}$ . Then the Gauss sum is  $g(\rho_7) = 2(\cos(2\pi/7) + e^{2\pi i/3} \cos(4\pi/7) + e^{4\pi i/3} \cos(6\pi/7))$  and we have

$$\begin{aligned} L_2((2, 4, 4, 2), (1, \rho_7, \rho_7, 1); B_2) &= \sum_{m, n=1}^{\infty} \frac{\rho_7(n) \rho_7(m+n)}{m^2 n^4 (m+n)^4 (m+2n)^2} \\ &= \frac{(-1)^{12+4}}{2^2 2!} \left( \frac{(2\pi i)^2}{2!} \right)^2 \left( \frac{(2\pi i)^4}{4! 7^4} g(\rho_7) \right)^2 \left( \frac{69967019}{6988350600} + \frac{102810289 \sqrt{-3}}{6988350600} \right) \\ &= g(\rho_7)^2 \pi^{12} \left( \frac{69967019}{181289027372537700} + \frac{102810289 \sqrt{-3}}{181289027372537700} \right). \end{aligned}$$

We give two more examples. Let  $\rho_5$  be the quadratic character of conductor 5. Then we have

$$\begin{aligned} L_2((2, 2, 2, 2), (\rho_5, \rho_5, \rho_5, \rho_5); B_2) &= \frac{92}{29296875} \pi^8; \\ L_3((2, 2, 2, 2, 2, 2), (\rho_5, \rho_5, \rho_5, \rho_5, \rho_5, \rho_5); A_3) &= -\frac{1856}{213623046875} \pi^{12}. \end{aligned}$$

The latter can be regarded as a character analogue of the formula in [1, Prop. 8.5].

## §11. Multiple Generalized Bernoulli Numbers

The generating function of multiple generalized Bernoulli numbers is given in terms of that of multiple Bernoulli polynomials as in the classical theory.

**Definition 5** (generating function [9, 10]). For  $\mathbf{t} = (t_\alpha)_{\alpha \in \Delta_+}$ ,

$$G(\mathbf{t}, \chi; X_r) = \sum_{\substack{a_\alpha=1 \\ \alpha \in \Delta_+}}^{f_\alpha} \left( \prod_{\alpha \in \Delta_+} \frac{\chi_\alpha(a_\alpha)}{f_\alpha} \right) F(\mathbf{f} \mathbf{t}, \mathbf{y}(\mathbf{a}; \mathbf{f}); X_r),$$

where  $F(\mathbf{t}, \mathbf{y}; X_r)$  is the generating function of multiple periodic Bernoulli functions and  $\mathbf{f} \mathbf{t} = (f_\alpha t_\alpha)_{\alpha \in \Delta_+}$ ,  $\mathbf{y}(\mathbf{a}; \mathbf{f}) = \sum_{\alpha \in \Delta_+} a_\alpha \alpha^\vee / f_\alpha$ .

**Definition 6** (multiple generalized Bernoulli numbers [9, 10]).

$$G(t, \chi; X_r) = \sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^{|\Delta_+|}} B_{\mathbf{k}, \chi}(X_r) \prod_{\alpha \in \Delta_+} \frac{t_\alpha^{k_\alpha}}{k_\alpha!},$$

$$B_{\mathbf{k}, \chi}(X_r) = \left( \prod_{\alpha \in \Delta_+} f_\alpha^{k_\alpha - 1} \right) \sum_{\substack{a_\alpha = 1 \\ \alpha \in \Delta_+}}^{f_\alpha} \left( \prod_{\alpha \in \Delta_+} \chi_\alpha(a_\alpha) \right) P(\mathbf{k}, \mathbf{y}(\mathbf{a}; \mathbf{f}); X_r).$$

cf.  $(X_r = A_1)$

$$G(t, \chi) = \sum_{a=1}^f \frac{\chi(a)}{f} F(ft, a/f) = \sum_{a=1}^f \frac{\chi(a)}{f} \frac{fte^{ft(a/f)}}{e^{ft} - 1} = \sum_{k=0}^{\infty} B_{k, \chi} \frac{t^k}{k!}.$$

$$B_{k, \chi} = f^{k-1} \sum_{a=1}^f \chi(a) B_k(\{a/f\}).$$

### §§11.1. Properties

**Theorem 6** ([9, 10]). Assume that  $f_\alpha > 1$  if  $\Delta$  is of type  $A_1$ . Then for  $w \in W$ ,

$$B_{w^{-1}\mathbf{k}, w^{-1}\chi}(X_r) = \left( \prod_{\alpha \in \Delta_+ \cap w^{-1}\Delta_-} (-1)^{k_\alpha} \chi_\alpha(-1) \right) B_{\mathbf{k}, \chi}(X_r).$$

Hence  $B_{\mathbf{k}, \chi}(X_r) = 0$  if there exists an element  $w \in W_{\mathbf{k}} \cap W_\chi$  such that

$$\prod_{\alpha \in \Delta_+ \cap w^{-1}\Delta_-} (-1)^{k_\alpha} \chi_\alpha(-1) \neq 1,$$

where  $W_{\mathbf{k}}$  and  $W_\chi$  are the stabilizers of  $\mathbf{k}$  and  $\chi$  respectively.

cf.  $(X_r = A_1)$

$$B_{k, \chi} = 0 \quad \text{if } (-1)^k \chi(-1) \neq 1.$$

Several other properties in the classical theory such as

$$F(t, y) = F(-t, -y) \text{ for } y \in \mathbb{R} \setminus \mathbb{Z}, \quad B_k(1-y) = (-1)^k B_k(y), \quad \frac{1}{t} \frac{\partial}{\partial y} F(t, y) = F(t, y)$$

can be reinterpreted in terms of root systems and Weyl groups.

## §12. Appendix: Integral Representation

The analytic continuations of multiple zeta-functions were already obtained by Matsumoto [11], Essouabri [3], de Crisenoy [2], etc. However we give yet another method which is a generalization of the formula

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\Gamma(s)(e^{2\pi i s} - 1)} \int_C \frac{z^{s-1}}{e^z - 1} dz \quad (C: \text{Hankel contour}).$$

For  $\xi \in \mathbb{C}^R$ ,  $a, s \in \mathbb{C}^N$  and  $b \in \mathbb{C}^{N \times R}$ , consider the multiple series

$$\zeta(\xi, a, b, s) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_R=0}^{\infty} \frac{e^{\xi_1 m_1} \cdots e^{\xi_R m_R}}{(a_1 + b_{11}m_1 + \cdots + b_{1R}m_R)^{s_1} \cdots (a_N + b_{N1}m_1 + \cdots + b_{NR}m_R)^{s_N}}.$$

**Theorem 7** ([4, 5]).

$$\zeta(\xi, a, b, s) = \frac{1}{\Gamma(s_1) \cdots \Gamma(s_N)} \prod_{t \in S} \frac{1}{e^{2\pi i t(s)} - 1} \times \int_{\Sigma} \frac{e^{(b_{11} + \cdots + b_{1R} - a_1)z_1} \cdots e^{(b_{N1} + \cdots + b_{NR} - a_N)z_N} z_1^{s_1-1} \cdots z_N^{s_N-1}}{(e^{z_1 b_{11} + \cdots + z_N b_{N1}} - e^{\xi_1}) \cdots (e^{z_1 b_{1R} + \cdots + z_N b_{NR}} - e^{\xi_R})} dz_1 \wedge \cdots \wedge dz_N,$$

where  $\Sigma$  is essentially a union of surfaces and  $S$  is a set of linear functionals on  $\mathbb{C}^N$ .

From the integrand, we can construct generating functions of Bernoulli numbers for nonpositive domain.

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